



# Entanglement Flavors

*Quaternion Analytics*

## QA Exploration series

*Quantum entanglement stands at the heart of modern physics, bridging realms once thought separate. In this exploration, we dive into entanglement from the perspective of quantum information, where qubits, the fundamental units of quantum data, intertwine across distances, reshaping our approach to computation and information processing.*

Marco Gambetta, Matteo Zangani

December 2, 2024

# Contents

|          |                                                                             |          |
|----------|-----------------------------------------------------------------------------|----------|
| <b>1</b> | <b>Theoretical Background on Entanglement</b>                               | <b>1</b> |
| <b>2</b> | <b>Entanglement in Quantum Information</b>                                  | <b>1</b> |
| 2.1      | Generalization to Multi-Qubit Systems . . . . .                             | 2        |
| <b>3</b> | <b>Creating Entanglement in Physical Quantum Computers</b>                  | <b>2</b> |
| 3.1      | Physical Mechanisms of Gate Implementations in Different Technologies . . . | 3        |
| 3.2      | Physical Steps for Entangling Qubits . . . . .                              | 4        |
| 3.3      | Error Correction and Coherence Maintenance . . . . .                        | 4        |
| <b>4</b> | <b>Entanglement Strength</b>                                                | <b>4</b> |
| 4.1      | Qubit Pair Scenarios . . . . .                                              | 5        |
| 4.2      | Analysis of Entanglement and Measurement Probabilities . . . . .            | 5        |
| 4.2.1    | Probabilities of Measurement in $ 0\rangle$ or $ 1\rangle$ . . . . .        | 5        |
| 4.2.2    | Entanglement Differences . . . . .                                          | 6        |
| 4.2.3    | Conclusion . . . . .                                                        | 6        |
| 4.3      | Effect of Post-Entanglement Rotations on Quantum Entanglement . . . . .     | 7        |
| 4.3.1    | Definition of Local Operations . . . . .                                    | 7        |
| 4.3.2    | Local Operations and Entanglement Preservation . . . . .                    | 7        |
| 4.3.3    | Impact on the Entangled State . . . . .                                     | 7        |

# 1 Theoretical Background on Entanglement

Entanglement is a uniquely quantum phenomenon in which the properties of particles become deeply interwoven, such that the state of one particle instantaneously affects the state of another, regardless of the distance separating them. This connection arises from the superposition principle in quantum mechanics, where particles exist in a range of possible states until they are observed. When two particles become entangled, their combined system adopts a single, unified quantum state that cannot be decomposed into independent states for each particle.

Physically, entanglement can be understood as a consequence of conservation laws, such as conservation of spin or momentum, applied at the quantum level. When particles interact under specific conditions, such as during collisions or when created simultaneously, their states become correlated in ways that persist even after they are separated. Measuring one particle's property (e.g., its spin) instantaneously determines the corresponding property in its entangled partner, reflecting a profound connection that transcends classical separability.

This phenomenon has been extensively tested, most famously in experiments validating Bell's theorem, which shows that entangled particles violate classical expectations of locality and realism. Entanglement underpins various quantum technologies, enabling applications that leverage its non-classical correlations for tasks such as secure communication, quantum computation, and sensing. The physical significance of entanglement extends beyond individual particles, illustrating a fundamental feature of quantum systems where information and properties are distributed across a network of particles rather than isolated within each one.

# 2 Entanglement in Quantum Information

In quantum information theory, entanglement is represented mathematically through the formalism of tensor products, where the state space of a multi-qubit system is described by the combined state vectors of individual qubits. For a pair of qubits, each with states in a two-dimensional Hilbert space  $\mathcal{H} = \mathbb{C}^2$ , the joint state of the two-qubit system is represented in the four-dimensional space  $\mathcal{H} \otimes \mathcal{H} = \mathbb{C}^4$ .

The simplest example of entanglement in a two-qubit system is the Bell state, which is a maximally entangled state. To understand the mathematical construction, consider two qubits in the computational basis states  $|0\rangle$  and  $|1\rangle$ . The state of a single qubit can be expressed as a superposition of these basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ . For a two-qubit system, the joint state can be written as a combination of tensor products of individual qubit states. If each qubit is initialized in  $|0\rangle$ , the combined initial state of the two-qubit system is  $|00\rangle$ , or in tensor notation,  $|0\rangle \otimes |0\rangle$ .

To create an entangled state, we apply a sequence of quantum gates. A common approach is to use a Hadamard gate on the first qubit, followed by a controlled-NOT (CNOT) gate, where the first qubit is the control and the second qubit is the target. Mathematically, these operations transform the initial state into an entangled state.

1. **Hadamard Gate:** The Hadamard gate  $H$  applied to a qubit in state  $|0\rangle$  creates an equal superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Applying this to the first qubit in the initial state  $|00\rangle$ , we obtain:

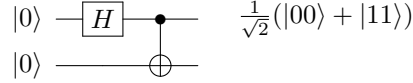
$$(H \otimes I)|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

2. **CNOT Gate:** Next, we apply the CNOT gate, defined by its action on the basis states. The CNOT operation flips the state of the target qubit (second qubit) if the control qubit (first qubit) is  $|1\rangle$ . When applied to  $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ , we obtain:

$$\text{CNOT} \left( \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The resulting state,  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , is a Bell state, one of the four maximally entangled two-qubit states. In this state, measurement of one qubit immediately determines the state of the other qubit, regardless of the physical separation between them.

The following circuit illustrates the process to create this entangled Bell state:



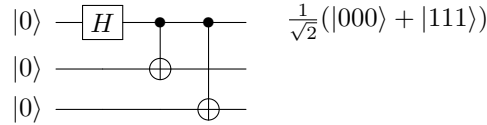
In this circuit: - The Hadamard gate ( $H$ ) is applied to the first qubit, creating a superposition of  $|0\rangle$  and  $|1\rangle$ . - The CNOT gate then entangles the two qubits, resulting in the Bell state  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , a maximally entangled state.

## 2.1 Generalization to Multi-Qubit Systems

Entanglement can be extended to systems with more than two qubits by applying combinations of Hadamard, CNOT, and other entangling gates, such as controlled-phase gates. For instance, applying additional CNOT gates in a three-qubit system can yield the GHZ state:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

The following circuit illustrates the creation of the GHZ state for three qubits:



In this circuit: - A Hadamard gate ( $H$ ) is applied to the first qubit, creating a superposition of  $|0\rangle$  and  $|1\rangle$ . - The first qubit is then used as the control qubit for two CNOT gates, entangling it with the second and third qubits. - This results in the GHZ state,  $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ , where all three qubits are maximally entangled.

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

which represents a maximally entangled state across all three qubits, where the state of each qubit is intrinsically tied to the state of the others.

Through controlled application of quantum gates, entanglement is systematically constructed in quantum information, forming the basis for many of its unique properties and functionalities. Each entangled state represents a quantum correlation that classical systems cannot replicate, and these correlations serve as building blocks for quantum operations that capitalize on entanglement's computational advantages.

## 3 Creating Entanglement in Physical Quantum Computers

In physical quantum computers, code instructions specify operations that must be translated into physical manipulations of qubit states. These manipulations are performed by

hardware-specific control systems, which interact with qubits using electromagnetic or optical signals to implement gate operations. While the mathematical representation of a gate is the same across different types of quantum computers, the physical processes vary significantly based on the qubit technology—whether superconducting, ion-based, or photonic.

### 3.1 Physical Mechanisms of Gate Implementations in Different Technologies

When the code specifies a gate operation, the quantum computer's control system initiates a sequence of physical actions to manipulate qubit states according to the desired transformation. Here's a deeper look at how each type of quantum computer executes basic gate operations.

**Hadamard Gate** The Hadamard gate, denoted  $H$ , is essential for creating superpositions. When applied to a qubit initially in the state  $|0\rangle$ , it places the qubit into an equal superposition state,  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , as represented on the Bloch sphere by a  $\pi/2$  rotation around an axis between the x- and z-axes.

1. **Superconducting Qubits:** In superconducting systems, qubits are controlled using microwave pulses, which drive oscillations in the qubit's energy levels. The Hadamard operation is achieved by precisely timing a microwave pulse to rotate the qubit's state by  $\pi/2$ . The pulse duration, frequency, and amplitude are carefully calibrated to produce the required rotation on the Bloch sphere, taking the qubit from  $|0\rangle$  to an equal superposition along the x-axis.

2. **Trapped Ion Qubits:** In ion-trap systems, the Hadamard gate is implemented using laser pulses. The ion's electronic states represent the qubit basis states, and a laser pulse with a specific frequency and phase creates a superposition by coherently exciting the ion's state. The pulse is calibrated to ensure the desired  $\pi/2$  rotation, creating a balanced superposition. Trapped ions are highly stable and offer high-fidelity Hadamard operations due to long coherence times and precise control over pulse parameters.

3. **Photonic Qubits:** For photonic qubits, the Hadamard gate often uses optical phase shifters and beam splitters. A beam splitter introduces a  $\pi/2$  phase shift, transforming an input photon's polarization into a superposition of horizontal and vertical polarizations. In this case, the Hadamard gate does not alter an energy level but manipulates the spatial or polarization path of the photon.

**CNOT Gate** The CNOT gate, a two-qubit gate, is more complex and varies significantly in implementation across different technologies. In a CNOT gate, the control qubit determines whether or not to flip the state of the target qubit.

1. **Superconducting Qubits:** The CNOT gate is implemented using a cross-resonance interaction, where one qubit is driven at the resonance frequency of the other. This technique induces a coupling between the two qubits, which, under specific pulse timing and frequency, creates a conditional rotation on the target qubit. The cross-resonance effect generates an effective Hamiltonian, often tuned using calibration pulses to implement the CNOT gate with high fidelity. Additionally, to reduce errors and improve gate fidelity, superconducting systems may use echo sequences to counteract unwanted noise.

2. **Trapped Ion Qubits:** For trapped ions, the CNOT gate is typically implemented using a shared motional mode that creates an indirect coupling between qubits. The ions are first cooled to their vibrational ground state, then laser pulses are applied to establish a coupling that enables conditional flipping of the target qubit based on the control qubit's state. This interaction is mediated by the ions' shared vibrational mode, and the CNOT gate is realized by carefully adjusting the timing and amplitude of the laser pulses to ensure the correct coupling strength.

3. **Photonic Qubits:** Photonic systems often implement the CNOT gate probabilistically due to the difficulty of strong two-qubit interactions. In linear optical systems, the

CNOT gate is realized using beam splitters, phase shifters, and single-photon detectors in combination with quantum interference. For example, by using a setup that includes polarizing beam splitters and waveplates, photonic systems can achieve a CNOT-like effect where the presence or absence of a photon in the control path determines the polarization rotation of the target photon.

### 3.2 Physical Steps for Entangling Qubits

To create entanglement, a quantum computer follows a precise sequence of physical actions in response to the code instructions. Let's take the example of creating a Bell state, which involves a Hadamard gate on one qubit followed by a CNOT gate, as an illustration of this process.

- **Initialization:** The qubits are initialized to the ground state  $|0\rangle$ , usually achieved by cooling in superconducting qubits or optical pumping in ion traps. For photonic qubits, initialization often involves generating photons in a specific polarization state.
- **Applying Hadamard Gate:** The Hadamard gate is applied to the first qubit, placing it in a superposition state. Depending on the technology, this involves either a microwave pulse (superconducting), a laser pulse (ion traps), or a beam splitter (photons), each creating the required  $\pi/2$  rotation to achieve the desired superposition.
- **Executing the CNOT Gate:** The CNOT gate then entangles the qubits. In superconducting systems, this involves tuning the cross-resonance frequency to induce the necessary Hamiltonian. In ion traps, laser pulses activate the shared vibrational mode between ions, enabling a conditional flip of the target qubit. In photonic systems, beam splitters and phase shifters are used along with photon detection to achieve a probabilistic entanglement, meaning that the gate succeeds only with a certain probability.
- **Resulting Entangled State:** After these operations, the qubits are now entangled. In the resulting Bell state,  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , the measurement outcome of one qubit will determine the state of the other, even if separated. This state forms the foundation for quantum algorithms and communication protocols that leverage entanglement.

### 3.3 Error Correction and Coherence Maintenance

Maintaining entanglement in a quantum computer requires sophisticated error correction and coherence-preserving techniques, as noise can rapidly disrupt the fragile entangled states. Superconducting systems use dynamic decoupling sequences, such as Carr-Purcell-Meiboom-Gill (CPMG) sequences, to mitigate decoherence effects. Trapped ions, benefiting from long coherence times, use active error correction schemes based on syndrome measurements to detect and correct errors. Photonic systems, while less susceptible to decoherence, employ specialized error-correcting codes to handle photon loss and detection inefficiencies.

Through these controlled operations and coherence-preserving measures, quantum computers transform abstract code instructions into precise, real-world operations, creating the entanglement needed for quantum computing tasks.

## 4 Entanglement Strength

In this section, we explore the behavior of entanglement when various quantum gates and rotations are applied to pairs of qubits. Specifically, we examine four scenarios involving pairs of qubits that undergo Hadamard gates ( $H$ ), controlled-NOT (CNOT) operations, and different types of rotations ( $R_z$  and  $R_y$ ).

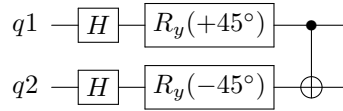
## 4.1 Qubit Pair Scenarios

We analyze four qubit-pair scenarios, each involving distinct combinations of quantum gates and rotations:

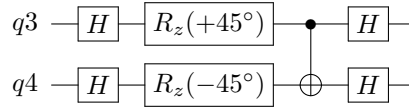
1. **Qubits  $q1$  and  $q2$ :** Applied through a Hadamard gate, rotated by  $R_y$  with angles  $+45^\circ$  and  $-45^\circ$  respectively, and then entangled with a controlled-NOT gate.
2. **Qubits  $q3$  and  $q4$ :** Applied through a Hadamard gate, rotated by  $R_z$  with angles  $+45^\circ$  and  $-45^\circ$  respectively, followed by a controlled-NOT gate, and concluded with a final Hadamard gate for measurement in the computational basis.
3. **Qubits  $q5$  and  $q6$ :** Applied through a Hadamard gate, entangled with a controlled-NOT gate, and then rotated by  $R_y$  with angles  $+45^\circ$  and  $-45^\circ$  respectively.
4. **Qubits  $q7$  and  $q8$ :** Applied through a Hadamard gate, entangled with a controlled-NOT gate, rotated by  $R_z$  with angles  $+45^\circ$  and  $-45^\circ$  respectively, and finally passed through a Hadamard gate.

Each of these scenarios can be represented by a quantum circuit, as illustrated below.

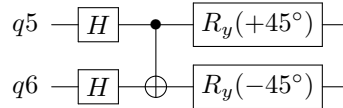
**Circuit for Qubits  $q1$  and  $q2$**



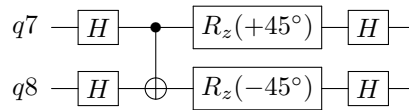
**Circuit for Qubits  $q3$  and  $q4$**



**Circuit for Qubits  $q5$  and  $q6$**



**Circuit for Qubits  $q7$  and  $q8$**



## 4.2 Analysis of Entanglement and Measurement Probabilities

### 4.2.1 Probabilities of Measurement in $|0\rangle$ or $|1\rangle$

For each scenario, the probability of measuring each qubit in  $|0\rangle$  or  $|1\rangle$  depends on the entanglement and the gates applied:

- **Qubits  $q1$  and  $q2$ :** The rotation  $R_y$  alters both amplitude and phase before the controlled-NOT gate, creating a robust entangled state with mixed probabilities for measuring either  $|00\rangle$  or  $|11\rangle$ .

- **Qubits  $q3$  and  $q4$ :** The  $R_z$  gate introduces a phase change without affecting amplitude, and the final Hadamard gate partially undoes the phase entanglement, yielding weaker correlations.
- **Qubits  $q5$  and  $q6$ :** The  $R_y$  rotation is applied post-entanglement, preserving a high entanglement level with complex phase-amplitude effects.
- **Qubits  $q7$  and  $q8$ :** The  $R_z$  rotation affects phase post-entanglement, and the final Hadamard gates lead to moderate correlations upon measurement.

#### 4.2.2 Entanglement Differences

Local rotations affect the correlation type and strength between qubits without destroying entanglement:

- **Qubits  $q1$  and  $q2$ :** Robust entanglement due to combined phase and amplitude changes.
- **Qubits  $q3$  and  $q4$ :** Primarily phase-based entanglement, yielding weaker correlations after final Hadamard gates.
- **Qubits  $q5$  and  $q6$ :** Strong entanglement with phase and amplitude effects from post-CNOT  $R_y$  rotations.
- **Qubits  $q7$  and  $q8$ :** Moderate entanglement with phase effects influenced by final Hadamard gates.

#### 4.2.3 Conclusion

This analysis illustrates how different gate arrangements and rotations affect entanglement and measurement probabilities in quantum circuits. Applying a local rotation after entanglement does not eliminate the entanglement but can change the nature of the correlation between qubits.

In comparing the two strongly entangled pairs,  $q1, q2$  and  $q5, q6$ , we observe that  $q5$  and  $q6$  exhibit slightly stronger entanglement. This difference is due to the order of operations within each circuit:

- For  $q5, q6$ , the CNOT gate is applied *before* the  $R_y$  rotations, meaning that the already-entangled state undergoes both amplitude and phase adjustments from the  $R_y(+45^\circ)$  and  $R_y(-45^\circ)$  rotations. These subsequent rotations add complexity to the entangled state, enhancing both amplitude and phase correlations and thereby strengthening the entanglement.
- For  $q1, q2$ , the  $R_y$  rotations are applied *before* the CNOT gate. Here, the CNOT operates on qubits that have undergone individual amplitude and phase adjustments, rather than entangling an already complex state. Consequently, the entanglement level, while strong, is slightly reduced relative to the  $q5, q6$  pair.
- Additionally, since the CNOT gate is the last major operation in the  $q5, q6$  circuit, the entangled state is measured almost immediately after entanglement is established. This positioning reduces the impact of time-dependent decoherence or dephasing effects that might otherwise weaken the correlations if more time were allowed before measurement. By minimizing the interval between entanglement and measurement,  $q5, q6$  maintains stronger correlations compared to  $q1, q2$ .

This contributes to the slightly stronger entanglement in  $q5, q6$  as compared to  $q1, q2$ .

### 4.3 Effect of Post-Entanglement Rotations on Quantum Entanglement

In quantum circuits, entanglement between qubits is typically established using a controlled operation, such as the CNOT gate, that correlates the states of two qubits. A common question arises when additional rotations or local operations are applied to entangled qubits after entanglement has been established: do these post-entanglement operations affect or destroy the entanglement?

#### 4.3.1 Definition of Local Operations

A *local operation* is an operation that acts on a single qubit (or a subset of qubits) independently of others in a multi-qubit system. Local operations include any single-qubit gate, such as the Pauli gates ( $X$ ,  $Y$ ,  $Z$ ), rotation gates ( $R_x$ ,  $R_y$ ,  $R_z$ ), and the Hadamard gate ( $H$ ). These gates transform the state of a single qubit without affecting others, making them incapable of creating or modifying entanglement between qubits. Local operations can change the phase or amplitude of an individual qubit's state, but they do not introduce correlations with other qubits.

In entangled systems, local operations do not alter the entanglement itself. For example, applying an  $R_y$  or  $R_z$  rotation to a single qubit in an entangled pair only changes the local properties of that qubit, leaving the entangled correlations intact.

#### 4.3.2 Local Operations and Entanglement Preservation

Entanglement is preserved under local operations, meaning that unitary operations like rotations applied to individual qubits do not eliminate the entanglement between them. When we apply rotations such as  $R_y$  or  $R_z$  to one or both qubits in an entangled pair, these operations modify the phase or amplitude of the individual qubits but do not disrupt the correlations established by the initial entangling gate.

#### 4.3.3 Impact on the Entangled State

Consider a maximally entangled Bell state, such as  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . If we apply a rotation  $R_y$  or  $R_z$  to one qubit in this entangled pair, the state remains entangled, though the specific form of the entanglement changes. The post-entanglement rotations transform the entangled state into a different correlated form, which can alter measurement probabilities but does not reduce the entanglement itself.

**Conclusion** Therefore, applying rotations to entangled qubits after the entangling gate does not destroy the entanglement. Instead, it alters the specific nature of the correlations, preserving the entanglement while creating a modified entangled state.

| Qubit Pair | Circuit                  | Entanglement Type     |
|------------|--------------------------|-----------------------|
| $q1, q2$   | Hadamard, CNOT           | Bell State            |
| $q3, q4$   | Hadamard, Rotation, CNOT | Partially Entangled   |
| $q5, q6$   | Hadamard, CNOT, Rotation | Complex Entanglement  |
| $q7, q8$   | Hadamard, CNOT           | Standard Entanglement |

Table 1: Basic Entanglement Types for Qubit Pairs